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MULTICRITERIAL DECISIONS WITH FUZZY WEIGHTS

1. Introduction

The pattern of multicriterial decision with fuzzy weights – the issue we are trying to solve here – received the following formulation:

Decident D has to hierarchise, complying with n criteria C_j; $j=\overline{1, n}$ an m number of variants (alternatives) V_i; $i=\overline{1, m}$, with fuzzy overall weights w_i; $i=\overline{1, m}$ (fuzzily) established on the basis of the estimations (appreciations, notes) made by s experts E^k; $k=\overline{1, s}$.

In order to grasp as accurately as possible the economic reality submitted to the decisional process the pattern should be completed by new elements. By introducing importance coefficients for the criteria and "authority" coefficients for the experts one gets a complex pattern of fuzzy multicriterial decision as an improvement to the similar determinist pattern [1]. The new one grasps the lack of precision implied by the estimations on the economic processes and phenomena submitted to analysis and optimization.

Among the methods expected to solve a fuzzy multicriterial decision pattern the following ones should be mentioned:

- the method of maximizing and minimizing sets [2];
- the method of membership level [3];
- the method of the average value of the rank [4];
- methods using artificial neural network [5, 6] etc.

The main disadvantages existing with these methods could be gathered as follows:

– the reckoning with unrealistic membership functions or fuzzy relationships;

– the usage of only one criterion and/or one expert alone;

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– the ampleness and complexity of the calculi implied by the above methods, which brings about many difficulties in implementing the methods.

In order to reduce some of the previously discussed disadvantages a method is displayed that is quite easy to apply. Its fuzzy weights associated to the alternatives are established through arithmetical operations with fuzzy numbers. The numbers are obtained by means of the membership functions created by the experts according to various estimation criteria. Then two linear and triangular membership functions and the absolute overall utility of lack variant are to be defined. On their basis the pattern's variant may reach the final optimus hierarchy.

2. Pattern of Multicriterial Decision with Fuzzy Weights (P.M.D.F.W.)

2.1. Membership functions of the variants

The decident aims to hierarchise m variants (alternatives) denoted by $V_i;\ i=\overline{1,m}$ complying with certain criteria and taking into account the experts opinions.

By using fuzzy numbers[1], one could define the normalized linear (trapezoidal) membership function of the V_i variant as $i=\overline{1,m}$, $\mu_{V_i}:\Re\to[0,1]$. The function was given by:

$$\mu_{V_i}(x)=\begin{cases}0, & x<\alpha_i\\ \dfrac{x-\alpha_i}{\beta_i-\alpha_i}, & \alpha_i\le x\le\beta_i\\ 1, & \beta_i<x<\gamma_i\\ \dfrac{\delta_i-x}{\delta_i-\gamma_i}, & \gamma_i\le x\le\delta_i\\ 0, & x>\delta_i\end{cases} \qquad (1)$$

[1] One calls fuzzy number the quartet $v=(\alpha,p,\gamma,\delta)$ or the pair $v=\left(\dfrac{\alpha}{\beta},\dfrac{\gamma}{\delta}\right)$, where $\alpha<\beta<\gamma<\delta<L$ are real numbers on a preference scale of L values linearly ordered.

The graphical representation of the $\mu_{V_i}(x)$ function is the $A_iB_iC_iD_i$ polygonal line displayed by Fig. 1 (trapezium without the big base).

There are also other ways of defining the normalized membership function of a variant. For instance, the A_iB_i and/or C_iD_i segments may be replaced by diagrams of some non-linear functions.

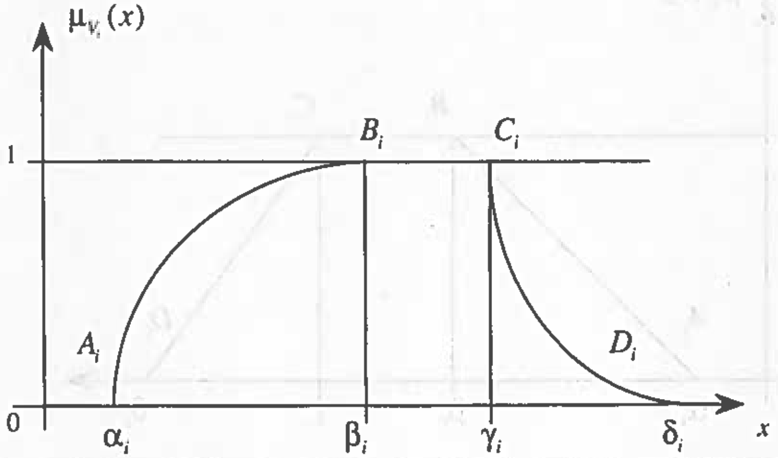

Fig. 1. (The Diagram) Graphical representation of the Linear Membership Function (1).

The type of the selected function has to be the result of a complex analysis made by every expert regarding the effects that the function values may have in establishing the final hierarchy.

The next example joins the two different tendences implied by the levels of preference expressed by the experts: one of low risk (concave function) and the other of high risk (convex function). Let there be the membership function $\mu_{V_i}: \Re \to [0,1]$ defined by:

$$\mu_{V_i}(x) = \begin{cases} 0, & x < \alpha_i \\ \dfrac{\ln(1 + x - \alpha_i)}{\beta_i - \alpha_i}, & \alpha_i \le x \le \beta_i \\ 1, & \beta_i < x < \gamma_i \\ \dfrac{e^{\gamma_i - x} - e^{\gamma_i - \delta_i}}{1 - e^{\gamma_i - \delta_i}}, & \gamma_i \le x \le \delta_i \\ 0, & x > \delta_i \end{cases} \tag{2}$$

Let us suppose that every E^k expert; $k=\overline{1,s}$ attaches the V_i alternative; $i=\overline{1,m}$, in relation to the C_j criterion $j=\overline{1,m}$ the following fuzzy number:

$$\tilde{V}_{ij}^{k}=\left(\alpha_{ij}^{k},\beta_{ij}^{k},\gamma_{ij}^{k},\delta_{ij}^{k}\right). \tag{3}$$

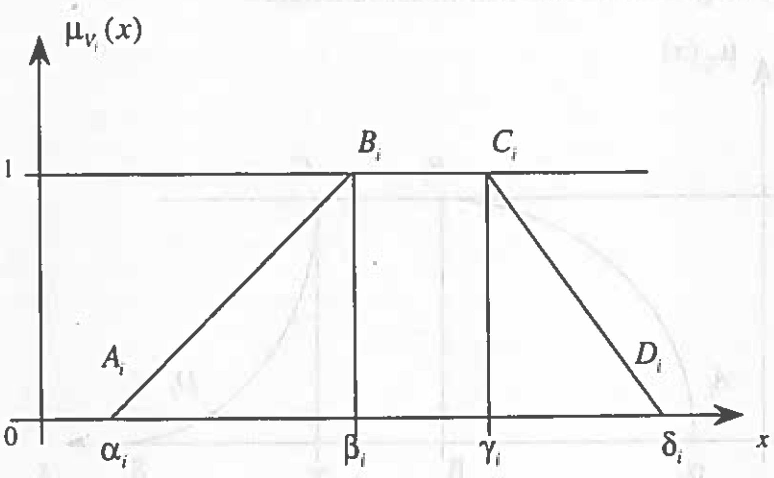

Fig. 2–Graphical representation of the Membership Function (2), Concave-Linear-Convex.

Actually this number measures the level of satisfaction that variant V_i provides for expert E^k in relation to criterion C_j.

By using the fuzzy numbers (3) the corresponding membership functions are established, as in case of both Ratioes (1) and (2). Hence for every criterion C_j; $j=\overline{1,n}$ and every variant V_i; $i=\overline{1,m}$ expert E^k; $k=\overline{1,s}$ defines the function $\mu_{V_iC_j}:\Re\to[0,1]$ given by:

$$\mu_{V_iC_j}^{k}(x)=\begin{cases}0, & x<\alpha_{ij}^{k}\\ \dfrac{x-\alpha_{ij}^{k}}{\beta_{ij}^{k}-\alpha_{ij}^{k}}, & \alpha_{ij}^{k}\le x\le\beta_{ij}^{k}\\ 1, & \beta_{ij}^{k}<x<\gamma_{ij}^{k}\\ \dfrac{\delta_{ij}^{k}-x}{\delta_{ij}^{k}-\gamma_{ij}^{k}}, & \gamma_{ij}^{k}\le x\le\delta_{ij}^{k}\\ 0, & x>\delta_{ij}^{k}\end{cases} \tag{4}$$

If $V_{ij}^k = \mu_{V_iC_j}(x),\ i = \overline{1, m};\ k = \overline{1, s}$ there follow the $M_1, M_2, \cdots M_n$ matrices of the form below:

$$M_j = \begin{pmatrix} v_{1j}^1 & v_{1j}^2 & \cdots & v_{1j}^k & \cdots & v_{1j}^8 \\ v_{2j}^1 & v_{2j}^2 & \cdots & v_{2j}^k & \cdots & v_{2j}^8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{ij}^1 & v_{ij}^2 & \cdots & v_{ij}^k & \cdots & v_{ij}^8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{mj}^1 & v_{mj}^2 & \cdots & v_{mj}^k & \cdots & v_{mj}^8 \end{pmatrix};\ j = \overline{1, n} \qquad (5)$$

2.2. *Membership function of criteria*

Every expert $E^k;\ k = \overline{1, s}$ estimates through the fuzzy number $\tilde{c}_j^k = (\varepsilon_j^k, \lambda_j^k, \eta_j^k, \theta_j^k);\ j = \overline{1, n}$, the importance he gives to criterion $C_j;\ j = \overline{1, n}$. By using these fuzzy numbers, expert E^k attaches to every criterion their linear (trapezoidal) normalized function $\mu_{C_j} : \Re \to [0,1]$ defined by:

$$\mu_{C_j}^k(x) = \begin{cases} 0, & x < \varepsilon_j^k \\ \dfrac{x - \varepsilon_j^k}{\lambda_j^k - \varepsilon_j^k}, & \varepsilon_j^k \le x \le \lambda_j^k \\ 1, & \lambda_j^k < x < \eta_j^k \\ \dfrac{\theta_j^k - x}{\theta_j^k - \eta_j^k}, & \eta_j^k \le x \le \theta_j^k \\ 0, & x > \theta_j^k \end{cases} \qquad (6)$$

If $C_j^k = \mu_{C_j}^k(x);\ j = \overline{1, n};\ k = \overline{1, s}$ there follows the matrix of the membership values of all the n criteria. The matrix is displayed below:

$$M=\begin{pmatrix} c_1^1 & c_1^2 & \cdots & c_1^k & \cdots & c_1^8 \\ c_2^1 & c_2^2 & \cdots & c_2^k & \cdots & c_2^8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_j^1 & c_j^2 & \cdots & c_j^k & \cdots & c_j^8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n^1 & c_n^2 & \cdots & c_n^k & \cdots & c_n^8 \end{pmatrix} \tag{7}$$

For the membership functions complying with those criteria the experts may also suggest other non-linear shapes (convex or concave) required by various factors: the experts competence level regarding the decisional matter and their level of information; the scale used in expressing their preferences; the nature of their criteria (qualitative and quantitative); their attitude toward risk, etc.

2.3. The Matrix of Fuzzy Overall Weights Associated to the Variants

The elements of the M_j matrices used; $j=\overline{1,n}$ and M given by Ratioes (5) and (7), it is expected that the fuzzy weights of the variants should be calculated complying with all the experts and regarding all the criteria. These weights are considered to be overall. Depending on the order in which the operations are made, one will get various methods to calculate the overall weights.

Next we propose a calculus method with its following steps:

a) For all the experts the formula below leads to the matrix of the fuzzy weights for the m variants and the n criteria:

$$p_{ij}=\frac{1}{s}\odot\left(v_{ij}^1\oplus v_{ij}^2\oplus\cdots\oplus v_{ij}^8\right);\ i=\overline{1,m};\ j=\overline{1,n}. \tag{8}$$

where $\oplus$ and $\odot$ stand for fuzzy addition and, respectively, fuzzy multiplication [7]. Let P by this matrix (with m rows and n columns) given by:

$$P=\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mj} & \cdots & p_{mn} \end{pmatrix} \tag{9}$$

b) For all the experts the next ratio leads to column matrix Q (m rows and one column) of the fuzzy weights corresponding to n criteria:

$$q_j = \frac{1}{s} \odot \left(c_j^i \oplus c_j^2 \oplus \cdots \oplus c_j^8 \right);\ j = \overline{1, n} \tag{10}$$

c) The matrix ratio below gives the column matrix W of the fuzzy overall weights of variants V_i, $i = \overline{1, m}$:

$$W = \frac{1}{n \cdot L} \odot (P \odot Q) \tag{11}$$

In Ratio (11) L stands for the number of the values of the scale used by the experts.

Other methods (leading to different results) imply as first step the fuzzy weights of variants V_i to be calculated for every expert. Then for all the experts the fuzzy weights averages as well as the overall weights are to be found out.

3. The Method of Overall Utility for Solving the P.M.D.F.W.

Let there by $W = (\alpha_i, \beta_i, \gamma_i, \delta_i);\ i = \overline{1, m}$ the fuzzy weights arisen from Ratio (11). The ordering method we suggest uses the weights and relies on the calculus of both lateral and overall utilities corresponding to every variant.

The steps of the method are the following:

E1. The next sums are reckoned:

$$\tilde{w}_i = \delta_i + \gamma_i - \beta_i - \alpha_i;\ i = \overline{1, m} \tag{12}$$

E2. The $\mu_{\tilde{w}_i}(x)$ membership functions (that may not be normalized) are derived from the ratioes:

$$\mu_{\tilde{w}_i}(x) = \begin{cases} 0, & x < \alpha_i \\ f_i(x), & \alpha_i \le x \le \beta_i \\ \tilde{w}_i, & \beta_i < x < \gamma_i \\ g_i(x), & x > \delta_i;\ i = \overline{1, m} \end{cases} \tag{13}$$

The functions $f_i(x)$ and $g_i(x)$ must be selected in such a way that $\mu_{\tilde{w}_i}(x)$ will be continue on $\Re$; moreover, $f_i(x)$ are strictly increasing functions while $g_i(x)$ are strictly decreasing.

E3. We find $\min\limits_{1\le i\le m} \tilde{w}_i = \tilde{w}_0$; $\min\limits_{1\le i\le m} \alpha_i = \alpha_0$; $\max\limits_{1\le i\le m} \delta_i = \delta_0$ and get the triangular linear membership functions (increasing and, decreasing respectively).

$$\overline{\mu}(x) = \begin{cases} \tilde{w}_0 \cdot \dfrac{x-\alpha_0}{\delta_0 - x_0}, & x \in [\alpha_0, \delta_0] \\ 0, & x \notin [\alpha_0, \delta_0] \end{cases}$$
$$\underline{\mu}(x) = \begin{cases} \tilde{w}_0 \cdot \dfrac{\delta_0 - x}{\delta_0 - \alpha_0}, & x \in [\alpha_0, \delta_0] \\ 0, & x \notin [\alpha_0, \delta_0] \end{cases} \quad (14)$$

E4. For every variants the lateral utilities follow from the formula:

$$U_s^i = \sup_x \{\mu_{\tilde{w}_i}(x) \cap \underline{\mu}(x)\};\ u_d^i = \sup_x \{\mu_{\tilde{w}_i}(x) \cap \overline{\mu}(x)\};\ i = \overline{1, m} \quad (15)$$

E5. The overall utilities are given by:

$$U_g^i = u_d^i + u_s^i;\ i = \overline{1, m} \quad (16)$$

E6. The variants are hierarchised according to the decreasing values of the U_g^i overall utilities.

4. Numerical Instance

4.1. P.M.D.F.W. Elements

a) Variants V_1, V_2, V_3 and V_4; criteria C_1, C_2 and C_3; experts E^1, E^2, and E^3 use a preference scale of 10 values linearly ordered. Thus $m = 4$; $n = 3$; $s = 3$; $L = 10$.

b) Matrix M_1, M_2 and M_3 corresponding to the C_1, C_2 and C_3 criteria are issued from Ratio (5):

$$M_1 = \begin{pmatrix} (6,7,8,0) & (4,5,5,6) & (6,7,8,9) \\ (6,7,7,8) & (4,4,5,5) & (0,1,2,2) \\ (5,5,6,6) & (7,8,8,9) & (4,5,5,5) \\ (7,8,8,9) & (2,2,2,2) & (4,5,5,6) \end{pmatrix} = \left(v_{i1}^k\right);\ i = \overline{1,4};\ k = \overline{1,3}$$

$$M_2 = \begin{pmatrix} (1,2,3,4) & (0,0,2,3) & (4,5,5,6) \\ (6,7,7,8) & (6,7,7,8) & (8,8,8,8) \\ (5,5,6,7) & (5,5,6,7) & (4,5,5,6) \\ (5,5,6,7) & (4,5,5,6) & (3,4,4,5) \end{pmatrix} = \left(v_{i2}^k\right);\ i = \overline{1,4};\ k = \overline{1,3}$$

$$M_3 = \begin{pmatrix} (4,5,6,7) & (7,8,8,9) & (1,2,3,4) \\ (3,4,4,5) & (7,8,8,9) & (4,5,5,6) \\ (7,8,8,8) & (2,3,3,4) & (6,7,7,8) \\ (1,2,3,4) & (7,8,8,8) & (6,7,7,8) \end{pmatrix} = \left(v_{i3}^k\right);\ i = \overline{1,4};\ k = \overline{1,3}$$

c) Matrix M given by Ratio (7):

$$M = \begin{pmatrix} (5,6,7,8) & (4,5,6,7) & (7,8,8,9) \\ (1,2,3,4) & (6,7,7,7) & (4,5,6,7) \\ (7,8,8,8) & (1,2,3,3) & (2,3,4,5) \end{pmatrix} = \left(c_j^k\right);\ j = \overline{1,3};\ k = \overline{1,3}$$

4.2. Matrix P, Q and W. Establishment

Ratioes (8), (10) and (11) used, we get the following matrices:

$$P = \frac{1}{3} \odot \begin{pmatrix} (16,19,21,24) & (5,7,10,13) & (12,15,17,20) \\ (12,12,14,15) & (20,22,22,24) & (17,17,17,20) \\ (16,18,21,21) & (14,16,17,20) & (15,18,18,20) \\ (13,15,15,17) & (12,14,15,18) & (14,17,18,20) \end{pmatrix};$$

$$Q = \frac{1}{3} \odot \begin{pmatrix} (16,19,21,24) \\ (11,14,16,18) \\ (10,13,15,16) \end{pmatrix};$$

$$W = \frac{1}{270} \odot \begin{pmatrix} (431,654,856,1130) \\ (550,757,901,1112) \\ (560,800,983,1184) \\ (479,702,825,1052) \end{pmatrix}$$

From matrix W 4 fuzzy overall weights of the 4 variants could be inferred:

$$w_1 = (1.60; 242; 3.17; 4.19) = (\alpha_1, \beta_1, \gamma_1, \delta_1)$$
$$w_2 = (2.04; 2.80; 3.34; 4.12) = (\alpha_2, \beta_2, \gamma_2, \delta_2)$$
$$w_3 = (2.07; 2.96; 3.64; 4.40) = (\alpha_3, \beta_3, \gamma_3, \delta_3)$$
$$w_4 = (1.77; 2.50; 3.06; 3.90) = (\alpha_4, \beta_4, \gamma_4, \delta_4)$$

4.3. *Application of the Calculus Method for Overall Utility*

E1. By Ratioes (12) one gets:

$\tilde{w}_1 = 3.34;\ \tilde{w}_2 = 2.62;\ \tilde{w}_3 = 3.00;\ \tilde{w}_4 = 2.73.$

E2. We consider $f_i(x)$ and $g_i(x)$ linear while constructing the $\mu_{\tilde{w}_i}(x)$ membership functions. For instance:

$$\tilde{w}_1(x) = \begin{cases} 0, & x < 1.60 \\ \tilde{w}_1 \dfrac{x - 1.60}{0.82}, & 1.60 \le x \le 2.42 \\ \tilde{w}_1, & 2.42 < x < 3.17 \\ \tilde{w}_1 \dfrac{4.19 - x}{1.02}, & 3.17 \le x \le 4.19 \\ 0, & x > 4.19 \end{cases}$$

where $\mu_{\tilde{w}_1} = 3.34$.

E3. $\min\limits_{1 \le i \le 4} \tilde{w}_1 = 2.62 = \tilde{w}_0;\ \min\limits_{1 \le i \le 4} \alpha_i = 1.60,\ \max\limits_{1 \le i \le 4} \delta_i = 4.40.$ The triangular linear membership functions given by (14) are as follows:

$$\overline{\mu}(x)=\begin{cases}2.62\cdot\dfrac{x-1.60}{2.80}, & x\in[1.6;4.4]\\ 0, & x\in[1.6;4.4]\end{cases}$$

$$\underline{\mu}(x)=\begin{cases}2.62\cdot\dfrac{4.4-x}{2.80}, & x\in[1.6;4.4]\\ 0, & x\in[1.6;4.4]\end{cases}$$

E4. Using the membership functions constructed at stages E2 and E3 and applying Ratio (15) one obtains the lateral utilities below:

$$u_s^1=2.133;\ u_s^2=1.722;\ u_s^3=1.719;\ u_s^4=1.982$$

$$u_d^1=1.880;\ u_d^2=1.843;\ u_d^3=2.132;\ u_d^4=1.675$$

For example the u_s^1 utility was reckoned this way:

$$\mu_{\tilde{w}_1}(x)=3.34\cdot\frac{x-1.60}{0.82}=\underline{\mu}(x)=2.62\cdot\frac{4.4-x}{2.80}\Leftrightarrow 2.338x-3.741=$$

$$=2.363-0.530x\Leftrightarrow x_s^1=2.12\Rightarrow\underline{\mu}(2.13)=2.62\cdot\frac{4.40-2.12}{2.80}=$$

$$=2.133\Rightarrow u_s^1=2.133.$$

E5. Complying with Ratioes (16) one gets the next overall utilities:

$$\text{U}_g^1=4.013;\ \text{U}_g^2=3.565;\ \text{U}_g^3=3.851;\ \text{U}_g^4=3.657.$$

E6. The hierarchy of the 4 variants depending on the C_1, C_2 and C_3 criteria and tallying with the E^1, E^2 and E^3 expert's estimations is like this: V_1, V_3, V_4, V_2.

Notes

a) If the absolute overall utilities are defined by $\text{U}_g^i=\left|u_d^i-u_s^i\right|$; $i=\overline{1,4}$ another hierarchy of variants V_1, V_3, V_4, V_2 appears.

b) Taking the overall utilities as the "areas" of some trapeziums given by the ratioes $\text{U}_g^i=\left|\left(u_d^i-u_s^i\right)\left(x_d^i-x_s^i\right)\right|$; $i=\overline{1,4}$ it follows that the variants setting in order is like this: V_1, V_3, V_4, V_2.

c) The way of defining the overall utilities and the average weights, may be as such the issue of a complex analysis for both the experts and the decident.

5. Conclusions

The methodology of constructing the pattern of multicriterial decision with fuzzy weights applies to the cases when the uncertainty called forth a series of reasons cannot be overlooked. We shall note only a few of such cases: the hierarchy of the Romanian counties tallying with various criteria of estimating their level of economic and social development; the selection of the optimum investment choice; the hierarchy of the variants regarding the environment protection etc.

The method suggested here is easy to apply and reduces some of the disadvantages implied by the methods mentioned in the beginning of the paper. Obtaining a hierarchy that complies with reality depends to a great extent on the accuracy when establishing the decisional criteria as well as the expert's and the decident's competence as regards the issue submitted to the mathematic and economic modelling.

References

1. Tacu, A.P., (sub redacţia), *Economia întreprinderii*, Ed. Academiei R.S.R., Bucureşti, 1973, pp. 81–103.
2. Jain, R., *A procedure for multiple-aspect decision making using fuzzy sets*, Int. I. Systems Sci. 8(1977), pp. 1–7.
3. Baas, S.M., Kwakernaak, H., *Rating and ranking of multiple-aspect alternatives using fuzzy sets*, Automatica 13(1990), pp. 47–57.
4. Campos, L.M., Gonzales, A., *A subjective approach for tanking fuzzy numbers*, Fuzzy Sets and Systems, 29(1988), pp. 145–156.
5. Cono, J.E., Delgado, M., Requena, I., *Using artificial neural networks to aid decision making process*, in: Proc. IWANN, Granada (Spain), Lecture Notes in Computer Science, Springer, Madrid, 1991, pp. 461–469.
6. Requena, I., Delgado, M., Verdegay, J.L., *Automatic ranking of fuzzy members with the criterion of decision-maker learnt by an artificial neural network*, Fuzzy Sets and Systems 64(1994), pp. 1–19.
7. Buckley, J.I., *Ranking alternatives using fuzzy members*, Fuzzy Sets and Systems, 15(1985), pp. 21–31.