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DETERMINATION OF OPTIMAL WAYS IN FUZZY GRAPHS

1. Classic-oriented graph

An impressive number of concrete economic issues can be solved out by using mathematic models based on graphs.

Many authors have dedicated whole chapters to the graph theory.

The **oriented graph** is defined in these works as an assembly G = (X, G) made of asset of knots called vertex $X=\{x_1, x_2,\ldots,x_n\}$ and an application G:X? P(X), which creates a correspondence of a subset of vertex to each vertex:

$$G = (X, \Gamma) \quad (1)$$

If a vertex x_j is an image of knot x_i by the application G ($x_j \in Gx_i$), the pair $u = (x_i, x_j)$ with the initial knot x_i and the final knot x_j is called arch.

The set of all arches of a graph is often marked with U and the graph is in this notation, the assembly:

$$G = (X, U) \quad (2)$$

To each arch $u \in U$ is associated a positive number l(u) called length. Practically, this number could represent, for instance, the distance between the two knots (localities) of the arch, or the cost of a transportation between these two localities, etc.

Thus, to a graph is associated a square matrix C with n rows and n columns with the following elements:

$$\begin{cases} l(x_i, x_j) & ,(x_i, x_j) \in U \\ \infty & ,(x_i, x_j) \notin U \\ 0 & ,i = j. \end{cases} \quad (3)$$

An. Inst. cerc. ec. „G. Zane", t. 10–11, Iaşi, 2001–2002, p. 69–77

In this definition it has been considered that the graph does not contain arches whose initial and final notes coincide (loops), and the lack of an arch from knot x_i to knot x_j was highlighted by attributing to the arch the improper value as costs ∞.

A **way d** from knot x_p to knot x_q is a succession of arches $u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_m$ where the initial knot of the arch u_1 is x_p, the final knot of the arch u_m is x_q, and the final knot of each arch u_k coincides with the initial knot of the next arch u_{k+1}.

The set of all ways from the knot x_p to the knot x_q will be marked D_{pq}.

The length of a distance **d** is the some of the arches lengths which is a component of the respective way:

$$l(d) = \sum_{u \in d} l(u) \qquad (4)$$

A ***road of a minimum length*** form the knot x_p to the knot x_q is a road $d^* \in D_{pq}$ for which:

$$l(d^*) = \min_{d \in D_{pq}} l(d) \qquad (5)$$

Example no 1.

Let us consider the graph with 5 vertex represented in Figure 1.

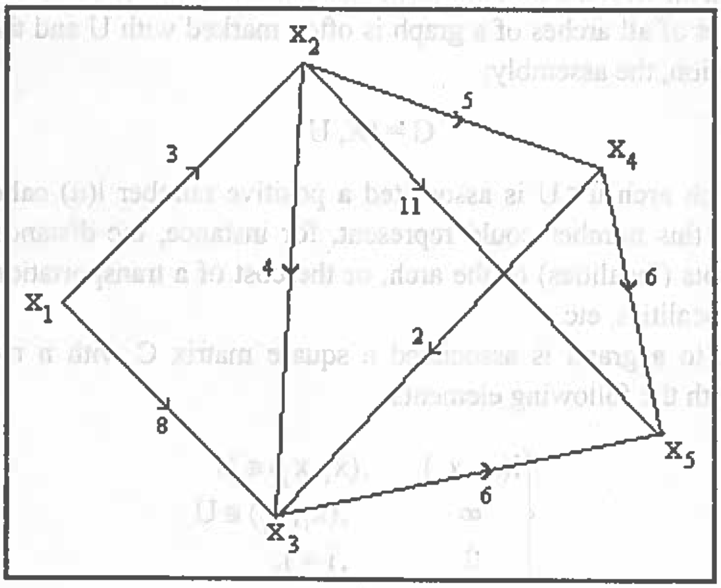

Figure 1. The oriented graph from example 1.

The 8 arches of the graph have the lengths given by the numbers within the figure on each arch.
The correspondent matrix of the graph is that from Table 1.

Table 1

The matrix associated to the graph from Figure 1

	x_1	x_2	x_3	x_4	x_5
x_1	0	3	8	∞	∞
x_2	∞	0	4	5	11
x_3	∞	∞	0	∞	6
x_4	∞	∞	2	0	6
x_5	∞	∞	∞	∞	0

Two graphs G_1 and G_2 having the same vertex number and the same arches but with different lengths are *equivalent graphs* if the optimal ways of the two graphs coincide.

The coincidence of the optimal ways refers to their routes and not to their lengths which can be different.

To determine the minimal way, in the graph theory, a large range of methods were given.

Out of all this, we will use the method bearing the name of the two authors Bellman-Kalaba in the hereby article.

2. Bellman-Kalaba algorithm for the determination of optimal ways

Let us consider the recursive succession $(v_i^{(k)})_{k=1,2,3,...}$ defined by the relations:

$$\begin{cases} v_i^{(1)} = c_{in} \\ v_i^{(k)} = \min\limits_{1\le j\le n}(v_j^{(k-1)} + c_{ij}) \end{cases}, i=\overline{1,n} \quad (6)$$

The elements of the succession are calculated recursively until the condition is fulfilled:

$$v_i^{(k)} = v_i^{(k+1)} \quad , \forall i = \overline{1,n} \quad (7)$$

The way of minimal length from vertex x_i to the final vertex x_n has the value $v_i^{(k)}$.

For **obtainning the optimal route** from vertex x_i to the vertex x_n it is first marked the initial vertex x_i.

If x_t is the last marked vertex of the route, the next vertex which has to be marked is one of the vertex x_s fulfilling the following conditions:

$$\begin{cases} x_s \in \Gamma x_t \\ v_t^{(k)} - c_{ts} = v_s^{(k)} \end{cases} \qquad (8)$$

The marking of the vertex of the optimal route is continued until a final vertex is reached x_n.

The route is made of the vertex marked in the order of their apparition.

It has to be underlined that the way of minimal length is not always solely determined.

By applying the algorithm presented previously to the graph from example 1, Table 1 will be filled in successively with 3 new rows whose elements are calculated according to relations (6) and (7).

The extended matrix obtained in this way is that from Table 2.

The 6th row $v_i^{(1)}$ (the first new row) is obtained by transposing the 5th column of the initial matrix.

The element on the i position from 7th row $v_i^{(2)}$ is the smallest element of the vector obtained by adding the previous row with row i of the initial matrix.

Thus, the first element of this row will be :

$$\min(\infty+0, 11+3, 6+8, 6+\infty, 0+\infty) = \min(\infty, 14, 14, \infty, \infty) = 14$$

Similarly, the 3rd element of the 8th row $v_i^{(3)}$ will be:

$$\min(14+\infty, 10+\infty, 6+0, 6+\infty, 0+6) = \min(\infty, \infty, 6, \infty, 6) = 6$$

Table 2

The extended matrix associated to the graph from Figure 1

	x_1	x_2	x_3	x_4	x_5
x_1	0	3	8	∞	∞
x_2	∞	0	4	5	11
x_3	∞	∞	0	∞	6
x_4	∞	∞	2	0	6
x_5	∞	∞	∞	∞	0
$v_i^{(1)}$	∞	11	6	6	0
$v_i^{(2)}$	14	10	6	6	0
$v_i^{(3)}$	13	10	6	6	0
$v_i^{(4)}$	13	10	6	6	0

The 9th row $v_i^{(4)}$ coincides with the 8th row $v_i^{(3)}$ and consequently the recursive succession is completely calculated, its final elements being those of the 8th row vector $v_i^{(3)}$.

The minimal distances from vertex x_1, x_2, x_3, x_4 and respectively x_5 to vertex x_5 will be 13, 10, 6, 6 and respectively 0.

For instance, for a determining the route of minimal length (13) from x_1 to x_5 we will mark vertex x_1 and then, each on its turn, the vertex fulfilling the condition (5.31), as follows:

$$Gx_1 = \{x_2, x_3\} \quad v_1^{(3)} - c_{12} = 13 - 3 = 10 = v_2^{(3)} \quad v_1^{(3)} - c_{13} = 13 - 8 = 5 \neq v_3^{(3)} = 6$$

we mark vertex x_2

$$Gx_2 = \{x_3, x_4, x_5\} \quad v_2^{(3)} - c_{23} = 10 - 4 = 6 = v_3^{(3)} \quad v_2^{(3)} - c_{24} = 10 - 5 = 5 \neq v_4^{(3)} = 6$$

$$v_2^{(3)} - c_{25} = 10 - 11 = -1 \neq v_5^{(3)} = 0$$

we mark vertex x_3

$$Gx_3 = \{x_5\} \quad v_3^{(3)} - c_{35} = 6 - 6 = 0 = v_5^{(3)}$$

we mark vertex x_5

The optimal way is: $d^*=(x_1, x_2, x_3, x_5)$ and its length is:

$$l(d^*)=3+4+6=13.$$

If we want to determine the ways of maximum length, the Bellman-Kalaba algorithm can be applied similarly with the exception of the replacing in the matrix attached to symbol ∞ with symbol $-\infty$ and the minimal operator with maximum operator.

3. The fuzzy graph. Determination of the optimal ways

In the nowadays' circumstances when the inflation has a growing rhythm, hard to be provisioned, the costs (lengths) associated to the arches of a graph can not be determined clearly, thus being more and more required the study of graphs from the fuzzy point of view in uncertain circumstances.

Considering the fuzzy numbers and the operations with them already known (see the article "Uncertainty modelling by fuzzy numbers" within the hereby publication) we will give the following definition:

A ***fuzzy graph*** is a graph for which the lengths of the arches $\tilde{l}(u)$ will be positive fuzzy numbers $\left(<\tilde{l}(u)>\ >0\right)$:

$$\tilde{G}=(X, U) \tag{9}$$

The length of a way d and the ways of minimal length d* for a fuzzy graph, are defined as the homologous notions from classic graphs:

$$\tilde{l}(d)=\sum_{u\in d}\tilde{l}(u) \tag{10}$$

$$\tilde{l}(d^*)=\min_{d\in D_{pq}}\tilde{l}(d) \tag{11}$$

The extended matrix of a fuzzy graph will have as elements the fuzzy numbers correspondent to the lengths of the arches, the real number 0 and symbol ∞, the operations with this symbol being the same with those from real numbers:

$$\left.\begin{array}{c}\tilde{a}+\infty=\infty\\ \tilde{a}<\infty\end{array}\right],\forall\tilde{a}\in NF(R) \tag{12}$$

To a fuzzy graph $\tilde{G}$ we will associate a classic graph $\langle\tilde{G}\rangle$ having as lengths of the arches, the real numbers associated to the fuzzy lengths $<\tilde{l}(u)>$. A highly important result is the one from the following theorem.

Theorem.

Any fuzzy graph is equivalent to its associated graph.

Demonstration. The result is quick, taking into account the defining manner of the order relation with fuzzy numbers. Thus, if d* is the optimal way in the fuzzy graph, this one is optimal in the associated graph, and viceversa, because:

$$\tilde{l}(d^*)=\min_{d\in D_{pq}}\tilde{l}(d)\Leftrightarrow\tilde{l}(d^*)\le\tilde{l}(d),\forall d\in D_{pq}\Leftrightarrow$$

$$\Leftrightarrow\left\langle\tilde{l}(d^*)\right\rangle\le\left\langle\tilde{l}(d)\right\rangle,\forall d\in D_{pq}\Leftrightarrow\left\langle\tilde{l}(d^*)\right\rangle=\min_{d\in D_{pq}}\left\langle\tilde{l}(d)\right\rangle$$

Since the optimal ways of the two graphs coincide, practically, the determination of the optimal way of a fuzzy graph is reduced to its determination for the associated classic graph.

Example no 2. Let us consider the graph of the previous example with the lengths of the arches presented in the figure below.

To the arch (x_2, x_5) the length 11 corresponds in the first example. This length was replaced because of the uncertainty with the triangular fuzzy number (11, 2, 4) which has the uncertain value 11, the extension to the left 2 and the extension to the right 4.

This fact symbolizes that the length of this arch can take values ranged between 11–2=9 and 11+4=15.

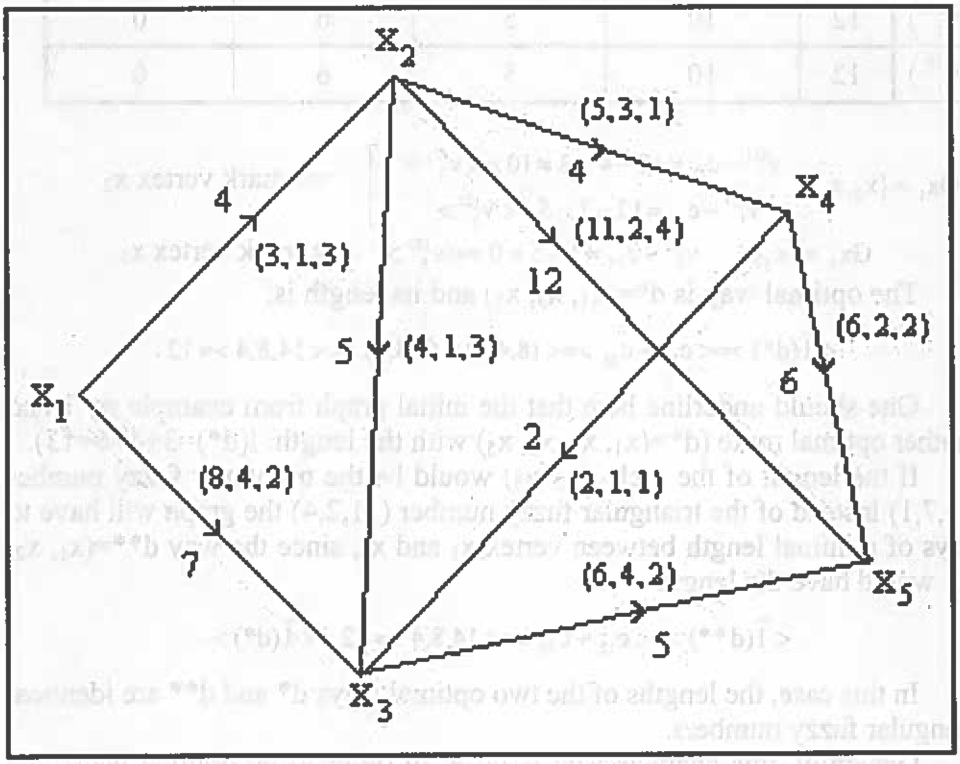

Figure 2. The fuzzy graph from example 2

The real number associated is: $< 11, 2, 4 >= 11 + \frac{4-2}{2} = 12$.

In Figure 2, on each arch is noted the triangular fuzzy number which represents its length, as well as, the associated real number.

By applying it to the graph associated to the Bellman-Kalaba algorithm is obtained the extended matrix from Table 3.

Table 3

The extended matrix associated to the associate graph from Figure 2

	x_1	x_2	x_3	x_4	x_5
x_1	0	<3,1,3>=4	<8,4,2>=7	∞	∞
x_2	∞	0	<4,1,3>=5	<5,3,1>=4	<11,2,4>=12
x_3	∞	∞	0	∞	<6,4,2>=5
x_4	∞	∞	<2,1,1>=2	0	<6,2,2>=6
x_5	∞	∞	∞	∞	0
$\langle v_i^{(1)} \rangle$	∞	12	5	6	0
$\langle v_i^{(2)} \rangle$	12	10	5	6	0
$\langle v_i^{(3)} \rangle$	12	10	5	6	0

$$Gx_1 = \{x_2, x_3\}: \quad \left.\begin{array}{l} v_1^{(2)} - c_{12} = 12 - 4 = 8 \neq 10 = < v_2^{(2)} > \\ v_1^{(2)} - c_{13} = 12 - 7 = 5 = < v_3^{(2)} > \end{array}\right] \quad \text{we mark vertex } x_3$$

$$Gx_3 = \{x_5\}: \quad v_3^{(2)} - c_{35} = 5 - 5 = 0 = < v_5^{(2)} > \quad \text{we mark vertex } x_5$$

The optimal way is $d^*=(x_1, x_3, x_5)$ and its length is:

$$< \tilde{l}(d^*) > = < c_{13} + c_{35} > = < (8,4,2) + (6,4,2) > = < 14,8,4 > = 12 .$$

One should underline here that the initial graph from example no 1 had another optimal route ($d^*=(x_1, x_2, x_3, x_5)$ with the length: $l(d^*)=3+4+6=13$).

If the length of the arch (x_2, x_5) would be the triangular fuzzy number (11,7,1) instead of the triangular fuzzy number (11,2,4) the graph will have to ways of minimal length between vertex x_1 and x_5, since the way $d^{**}=(x_1, x_2, x_5)$ would have the length:

$$< \tilde{l}(d^{**}) > = < c_{12} + c_{25} > = < 14,8,4 > = 12 = < \tilde{l}(d^*) > .$$

In this case, the lengths of the two optimal ways d^* and d^{**} are identical triangular fuzzy numbers.

Generally, this characteristic is false. In order to be optimal ways, the ways need to have real numbers associated to lengths, equal, but it is not necessary for the two lengths to be identical fuzzy numbers.

None of the two optimal ways from example 2 have the route of the optimal way of the initial graph from example 1.

The optimal route of the initial graph in comparison with the two optimal routes of the fuzzy graph differs by the number of arches which make the respective routes: 3 arches for the initial graph, and for the fuzzy graph, in both cases, 2 arches for each.

One should underline that the method approached in this example is not dependent on the choice as lengths of some triangular fuzzy numbers, following the same course for any choice of fuzzy numbers (bell, square, etc.).

References

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