SOLUTIONS IN THE BARGAINING GAMES

1. Introduction

The optimum solutions and winnings of the cooperative game players, games that allow, due to bargaining, agreement conclusions, mixed strategy correlation or utility transfer from one player to another, may be determined in different ways.

The purpose of our study is the presentation of these axiomatic solutions for bargaining games, accompanied by some observations.

In a two-people game with arbitrary sum (like the bargaining game) there is a certain subset S of the Euclidean plane $\mathbf{R}^2$, called admissible set, with the characteristic that, for any $(u, v) \in S$, two players may obtain the utilities (winnings) u and v. However, this means cooperation through utility transfer between players or acceptance of a cooperation price. The minimum level of the winning that a player may accept during the bargaining process, is the maximinimum value of the game. [1]

Given the set S and the maximinimum values (u*, v*), we must find rules that associate the triplet (S, u*, v*) with a solution obtained through bargaining, called $(\bar{u}, \bar{v}) = f(S, u^*, v^*)$. Although the winning depends on the personality and bargaining skills of each player, one must set a sufficient number of axioms that guarantee the existence of a unique function f for any bargaining of the game given by the function f must meet.

In bargaining theory, as in much of cooperative game theory, the physical outcomes involved in the bargaining process are ignored and only the resulting cardinal utility combinations of the players are considered. In other words, the theory assumes that any two bargaining situations are the same if they yield the same set of feasible utility combinations.

In this study we survey some of the axiomatic solutions that have been suggested in this theory and some of their properties. Our goal is to keep the presentation simple, unified and without the strangest mathematical versions of the theorems. A very comprehensive coverage of the literature dealing with this theory can be found in [2].

An. Inst. cerc. ec. „G. Zane", t. 10–11, Iaşi, 2001–2002, p. 79–84

2. Solutions in the bargaining games

Formally we describe a 2-person bargaining game by a pair (t, S) where $t \in R^2$ and $S \subseteq R^2$. We assume that the pair (t, S) satisfies the following conditions:

i) $t \in S$;

ii) S is compact and convex;

iii) There is at least one $p \in S$ with $p > t$, $(p_i > t_i \text{ for } i = 1,2)$.

The assumption that S is convex is reasonable in many applications and certainly when the player utilities are of the von Neumann-Morgenstern (V-M) type. We let N be the set of all bargaining games satisfying these three conditions.

The elements of S, the feasible set, are the utility pairs that the players can receive under cooperation if they reach a unanimous agreement. The disagreement point t (sometimes referred to as threat point) is the utility pair that the player have for the state of "negotiations failed, proceed without attempting to reach unanimity".

More precise interpretations of S and t depend on the particular situation that is being modeled. For example when the two bargainers represent a seller and a buyer of a certain item we may let t stand for the utilities of no exchange. S then represents all the feasible utilities that arise from all the possible exchanges between them.

The axiomatic approach proves to be very useful, since it succeeds in choosing a unique solution through a small number of simple conditions. Whatever this process is, the players will end up at our solution if our axioms are correct for their behavior. In the case of arbitration the proposed axioms give the arbitrator a rational on which he is basing his decision.

Definition 1: Given a bargaining pair (t, S) and a point $p \in R^2$, we say that p is ***individually rational*** if $p \geq t, (p_i > t_i \text{ for } i = 1,2)$.

Definition 2: We say that p is ***Pareto optimal*** if $p \in S$ and for every $q \in S$ if $q \geq p$ then $q = p$.

Definition 3: A ***solution*** is a function $\varphi : N \rightarrow R^2$ such that for every $(t,S) \in N$, $\varphi(t,S) \in S$.

Definition 4: A solution φ is called ***symmetric*** if for every $(t,S) \in N$,
$f(\varphi(t,S)) = \varphi(f(t), f(S))$,

where $f(S) = \{f(p) \mid p \in S\}, f : R^2 \to R^2, f(x,y) = f(y,x)$.

2.1. The Nash Solution

The ***Nash solution*** [3] is the function $\varphi : N \to R$ which select the individually rational utility pair with a maximal ***Nash product***, $(p_1 - t_1)\cdot(p_2 - t_2)$. Formally, for every bargaining pair (t,S), $\varphi(t,S)$ is the individually rational utility pair with the property that for every individually rational feasible utility pair $(q_1, q_2) \in S$:

$$[\varphi_1(t,S) - t_1]\cdot[\varphi_2(t,S) - t_2] \geq (q_1 - t_1)\cdot(q_2 - t_2) \qquad (1)$$

Thus the objective of the Nash solution is to maximize the product of the utility gains of the players. This maximization takes place over the individually rational outcomes.

It is easy to check that the maximum of the Nash product is attained at a unique point since the feasible set is convex. Thus $\varphi(t,S)$ is a unique feasible point for every bargaining pair $(t,S) \in N$.

Definition 5: We say that a solution is ***independent of irrelevant alternatives*** if for every two bargaining pairs (t,S) and (t,T) with $S \subseteq T$, if $\varphi(t,S) \in S$ then $\varphi(t,S) = \varphi(t,T)$.

Definition 6: We say that a solution to the bargaining problem is ***invariant under affine transformations of utility scale*** if for every player i, for every bargaining pair (t,S) and for every affine transformation of utility scale T_i we have:

$$T_i(\varphi(t,S)) = \varphi(T_i(t,S)) \text{ for } i = 1,2 \qquad (2)$$

Theorem 1: (Nash) A solution is Pareto optimal, symmetric, independent of irrelevant alternatives and invariant under affine transformations of utility scale if and only if is the Nash solution.

It is easy to see that the Nash solution satisfies these four properties. Thus if we accept that a solution should satisfy these conditions we must adopt the Nash solution and only it as our choice.

2.2. The Kalai-Smorodinsky Solution [4]

For every bargaining pair $(t,S) \in N$ we define the ***ideal point*** I of the pair by:

I_1=Max{p_1 / for some $p_2 \in R$, (p_1,p_2) is an individually rational feasible point in (t,S)}

I_2=Max{p_2 / for some $p_1 \in R$, (p_1,p_2) is an individually rational feasible point in (t,S)}

The Kalai-Smorodinsky (KS) solution is the function f that choose for every bargaining pair (t,S) the unique Pareto optimal point (p_1, p_2) with:

$$(p_1 - t_1)\cdot(I_2 - t_2) = (p_2 - t_2)\cdot(I_1 - t_1) \qquad (3)$$

We can supply an axiomatic rationale for the (KS) solution as we did for the Nash solution. Here we would not accept the independence of irrelevant alternatives condition. We adopt instead a condition of individual monotonicity.

Definition 7: For every bargaining pair (t,T) we say that p_2 is ***a rational demand for player 2*** if there is a pair (p_1, p_2) wich is feasible and individually rational in (t,S).

We say that the bargaining pair (t,W) is better ***for player 1*** than the bargaining pair (t,S) if the rational demands of player 2 are the same in both pairs, and for every such rational demand p_2 we have:

$$\sup\{p_1 / (p_1,p_2) \in W\} \geq \sup\{p_1 / (p_1,p_2) \in S\} \qquad (4)$$

We say that a solution f is ***individually monotonic*** for player 1 if whenever (t,W) is better for him than (t,S) then $f_1(t,W) \geq f_1(t,S)$.

Definition 8: A solution f is ***individually monotonic*** if the same property holds for both players.

We have now:

Theorem 2: A solution is symmetric, Pareto optimal, invariant under affine transformations of utility scale and individually monotonic if and only if it is the Kalai-Smorodinsky solution.

2.3. The Utilitarian Solution [5]

In this part of our study we implicity assume that every one of our players is using the same utility function with the same scale as we vary the bargaining pairs under consideration.

A solution will be called ***utilitarian*** if there are weights $\lambda = (\lambda_1, \lambda_2) \in R_+^2$ such that for every bargaining pair (t,S), we have:

$$f(t,S) = \max[\lambda_1(p_1 - t_1) + \lambda_2(p_2 - t_2)] = \max(\lambda_1 p_1 + \lambda_2 p_2) \qquad (5)$$

where the maximization takes place over the pairs (p_1, p_2) which are feasible individually rational elements of S.

In (5) the weights λ_1, λ_2 should be assigned to the utility scales of the two players, and then for every bargaining pair we would be maximizing the weighted sum of the utilities.

Notice that for a given utilitarian solution, with weight (λ_1, λ_2), if the scales that the players use were changed by defining new utility functions $\overline{p_i} = \lambda_i p_i$ for i=1,2, then the objective of the utilitarian solution is to always choose the outcome that maximizes the symmetric sum of the gains in units of $\overline{p_i}$.

2.4. The Egalitarian Solution [6]

In this selection we restrict our attention to a subset $N_0 \subseteq N$ of bargaining pairs $(t, S) \in N$ that satisfy the following additional properties:

1. For every $x \in S, x \geq t$, i.e., S consists only of individually rational outcomes;
2. Free disposal of utility, if $x \in S$ and $t \leq y \leq x$ then $y \in S$;
3. Existence of small utility transfers, if $t < (p_1, p_2) \in S$ then there is a pair $(q_1, q_2) \in S$ with $q_1 > p_1$ and there is a pair $(r_1, r_2) \in S$ with $r_2 > p_2$.

A solution f is called ***egalitarian*** if there are weights $\lambda_1, \lambda_2 > 0$ such that for every $(t, S) \in N_0$, $f(t, S)$ is Pareto optimal in S and satisfies:

$$\lambda_1 \cdot (f_1(t, S) - t_1) = \lambda_2 \cdot (f_2(t, S) - t_2) \tag{6}$$

Let be f an egalitarian solution with some fixed weights λ_1 and λ_2. We can easily check that it satisfies definitions 5 and 8.

We define a weak version of the scale invariance condition that egalitarian solutions do satisfy.

Definition 9: A solution f is ***invariant under translations*** if for every $a \in R^2$ and every $(t, S) \in N_0$ we have:

$$f(a+t, a+S) = a + f(t, S) \tag{7}$$

The invariance under translations guarantees that if each of the players receives a fixed prize regardless of reaching agreement and independently of the bargaining process, the prize will not effect the outcome of the bargaining.

References

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