# UNCERTAINTY MODELLING BY FUZZY NUMBERS

## 1. Fuzzy numbers, models of the uncertain information

A set of real numbers is well-defined by its characteristic function:

$$f_A : \mathbf{R} \longrightarrow \{0,1\} \quad (1)$$

Value 0 corresponds to real numbers not belonging to set A and value 1 symbolizes the belonging to the set.

A fuzzy set in **R**, $A \in \mathbf{F}(\mathbf{R})$, will have the characteristic function with values ranging within the whole interval [0,1]:

$$\mu_A : \mathbf{R} \longrightarrow [0,1] \quad (2)$$

An intermediary value between non-belonging 0 and belonging 1 symbolizes the degree (of safety) of the belonging to set A.

A ***fuzzy number***, ($\tilde{a} \in \mathbf{NF}(\mathbf{R}) \subset \mathbf{F}(\mathbf{R})$), represents a fuzzy set in **R** with the following properties:

– has as a nucleus a point or a closed interval:

$$N(\tilde{a}) \overset{def.}{=} \{x \in R \mid \mu_{\tilde{a}}(x) = 1\} = [a^m, a^M],\ a^m \le a^M \quad (3)$$

– on each of the left and right portions of the nucleus, the characteristic function is continuous and increasing, respectively decreasing;

– the characteristic function is integrable on **R** (and the graph of the characteristic function has the axis of the abscissas as horizontal asymptote to $\pm\infty$).

The support of a fuzzy number is an open interval (s, d) finite or not to one or both ends:

$$S(\tilde{a}) = \{x \in R \mid \mu_{\tilde{a}}(x) > 0\} \quad (4)$$

The positive numbers $a^s = a^m - s$ şi $a^d = d - a^M$ , $a^s, a^d \in \overline{R}_+$ are called the elongation to left and respectively the elongation to right.

Thus, a fuzzy number is a system made of five elements $\tilde{a} = (a^m, a^M; a^s, a^d, \mu_{\tilde{a}})$ with the following properties:

$$
\left.
\begin{array}{l}
, a^m \le a^M \in R \quad , a^s, a^d \in \overline{R}_+ \\
\mu_{\tilde{a}}(x) = \begin{cases} \mu^s_{\tilde{a}}(x) & , a^m - a^s < x < a^m \\ 1 & , a^m \le x \le a^M \\ \mu^d_{\tilde{a}}(x) & , a^M < x < a^M + a^d \\ 0 & , x \notin (a^m - a^s, a^M + a^d) \end{cases} , \mu^s_{\tilde{a}}(x), \mu^d_{\tilde{a}}(x) \in (0,1) \\
\mu^s_{\tilde{a}}(x') < \mu^s_{\tilde{a}}(x'') \quad , \forall x', x'' \ni \ a^m - a^s < x' < x'' < a^m \\
\mu^d_{\tilde{a}}(x') > \mu^d_{\tilde{a}}(x'') \quad , \forall x', x'' \ni \ a^M < x' < x'' < a^M + a^d \\
\lim_{\substack{x \to a^m - a^s \\ x > a^m - a^s}} \mu_{\tilde{a}}(x) = 0, \quad \lim_{\substack{x \to a^M + a^d \\ x < a^M + a^d}} \mu_{\tilde{a}}(x) = 0; \\
\exists \int_{-\infty}^{+\infty} \mu_{\tilde{a}}(x)dx = \lim_{t \to \infty} \int_{-t}^{+t} \mu_{\tilde{a}}(x)dx \in R
\end{array}
\right\} \quad (5)
$$

The nucleus, as a compact interval, can be made of one point ($a^m = a^M$, degenerated interval), case in which $\tilde{a}$ is called ***sharp fuzzy number***.

Opposingly ($N(\tilde{a}) = [a^m, a^M]$, $a^m < a^M$), $\tilde{a}$ is ***flat fuzzy number***.

The support is a finite or not interval, to one or both ends.

The future economic information is more or less uncertain.

If, for instance, taking into account the calculation of the inflation average of the past two years, one can establish as far as the supply with raw materials for the next trimester is concerned, a monetary fund need of 1000 u.m., it is very unlikely for this information to be the real one. Most likely it is an amount ranging within –5% and +10%. This amount is well represented by a triangular fuzzy number (according to the description of the next paragraph) $\tilde{a} = (1000, 50, 100)$ with the following belonging function:

$$
\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - 950}{50} & , 950 < x \le 1000 \\ \dfrac{1100 - x}{100} & , 1000 < x < 1100 \\ 0 & , x \notin (950, 1100). \end{cases}
$$

## 2. Classes of fuzzy numbers

Varying with the type of analytical expressions of the belonging functions, the fuzzy numbers can be grouped into different sub-classes.

In practice the most frequently used are the triangular and trapezoidal fuzzy numbers .

A ***fuzzy trapezoidal number*** $\tilde{a} = (a^m, a^M, a^s, a^d) \in \mathbf{Tp(R)}$, (Figure 1.a) has the following characteristic function:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a^m}{a^s} + 1 & , a^m - a^s < x < a^m \\ 1 & , a^m \le x \le a^M \\ \dfrac{a^M - x}{a^d} + 1 & , a^M < x < a^M + a^d \\ 0 & , x \notin (a^m - a^s, a^M + a^d). \end{cases} \quad (6)$$

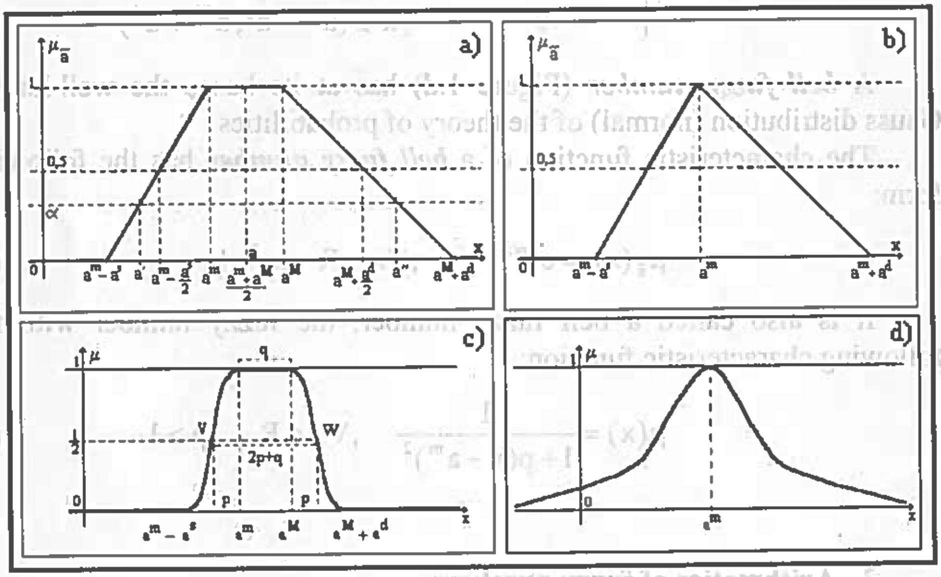


*Figure 1.* Fuzzy numbers: a) trapezoidal b) triangular c) square d) bell.

A sharp fuzzy trapezoidal number ($a^m = a^M$) is called ***fuzzy triangular number*** being symbolized by a third form $\tilde{a} = (a^m, a^s, a^d)$ (Figure 1.b).

The degenerated triangular fuzzy number $(a^m, 0, 0)$ corresponds to the real number $a^m$.

A ***square fuzzy number*** (Figure 1.c) has the restriction of the characteristic function on the two portions (intervals) of different monotonies, made of parabolas:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{(x - a^m + a^s)^2}{2(a^s - p)^2} & , a^m - a^s < x < a^m - p \\ 1 - \dfrac{(x - a^m)^2}{2p^2} & , a^m - p \le x < a^m \\ 1 & , a^m \le x \le a^M = a^m + q \\ 1 - \dfrac{(x - a^M)^2}{2p^2} & , a^M < x < a^M + p \\ \dfrac{(x - a^M - a^d)^2}{2(a^d - p)^2} & , a^M + p < x < a^M + a^d \\ 0 & , x \notin (a^m - a^s, a^M + a^d) \end{cases} \quad (7)$$

A ***bell fuzzy number*** (Figure 1.d) has at its bases the well-known Gauss distribution (normal) of the theory of probabilities.

The characteristic function of a ***bell fuzzy number*** has the following form:

$$\mu_{\tilde{a}}(x) = e^{-p(x-a^m)^2} \quad , \forall x \in R \quad , p > 1 \quad (8)$$

It is also called a bell fuzzy number, the fuzzy number with the following characteristic function:

$$\mu(x) = \frac{1}{1 + p(x - a^m)^2} \quad , \forall x \in R \quad , p > 1 \quad (9)$$

## 3. Arithmetics of fuzzy numbers

Taking into account that fuzzy numbers represent mathematic models of the uncertain numeric information, one can easely raise the issue of operation definition with these numbers.

Taking two fuzzy numbers: $\widetilde{a}=(a^m, a^M, a^s, a^d, \mu_{\widetilde{a}})$ and $\widetilde{b}=(b^m, b^M, b^s, b^d, \mu_{\widetilde{b}})$.

***The addition of the fuzzy numbers*** $\widetilde{a}+\widetilde{b}$ is defined as follows:

$$\left.\begin{array}{l} \widetilde{a}+\widetilde{b}=(a^m+b^m, a^M+b^M, a^s+b^s, a^d+b^d, \mu_{\widetilde{a}+\widetilde{b}}) \\ \mu_{\widetilde{a}+\widetilde{b}}(x)=\begin{cases} \left[\mu s_{\widetilde{a}}^{-1}+\mu s_{\widetilde{b}}^{-1}\right]^{-1}(x) & ,x\in(s_1,d_1) \\ \left[\mu d_{\widetilde{a}}^{-1}+\mu d_{\widetilde{b}}^{-1}\right]^{-1}(x) & ,x\in(s_2,d_2) \\ 1 & ,x\in[d_1,s_2] \\ 0 & ,x\notin(s_1,d_2) \end{cases} \\ \text{where}: d_1\overset{not.}{=}a^m+b^m,\quad s_1\overset{not.}{=}d_1-a^s-b^s, \\ s_2\overset{not.}{=}a^M+b^M,\quad d_2\overset{not.}{=}s_2+a^d+b^d \end{array}\right] \quad (10)$$

***The multiplication of a fuzzy number*** $\widetilde{a}=(a^m, a^M, a^s, a^d, \mu_{\widetilde{a}})$ ***with a scalar*** $t\in R$ makes a new fuzzy number $t\widetilde{a}$ defined as follows:

$$\left.\begin{array}{l} t\widetilde{a}=\begin{cases} (ta^m, ta^M, ta^s, ta^d, \mu_{t\widetilde{a}}) & ,t>0 \\ (ta^M, ta^m, -ta^d, -ta^s, \mu_{t\widetilde{a}}) & ,t<0 \\ (0,0,0,0,\mu_{\widetilde{0}}) & ,t=0 \end{cases} \\ \text{where}: \quad \mu_{t\widetilde{a}}(x)=\mu_{\widetilde{a}}\left(\frac{x}{t}\right),\forall x\in R \;,(\forall)t\in R\setminus\{0\} \end{array}\right] \quad (11)$$

***The opposite of the fuzzy number*** $\widetilde{a}$ marked $-\widetilde{a}$ is obtained as follows:

$$-\widetilde{a}\overset{def}{=}(-1)\cdot\widetilde{a}=(-a^M,-a^m,a^d,a^s,\mu_{-\widetilde{a}}),\quad \mu_{-\widetilde{a}}(x)=\mu_{\widetilde{a}}(-x)\quad ,\forall x\in R \quad (12)$$

The addition of the fuzzy numbers is associative and commutative and the triangular fuzzy number (real) $\widetilde{0}$=(0,0,0) is a neutral element for addition. Generaly, the addition between a fuzzy number and its opposite does not have as result $\widetilde{0}$:

$$\left.\begin{array}{l} \left.\begin{array}{c} (\widetilde{a}+\widetilde{b})+\widetilde{c}=\widetilde{a}+(\widetilde{b}+\widetilde{c}) \\ \widetilde{a}+\widetilde{b}=\widetilde{b}+\widetilde{a} \\ \widetilde{a}+\widetilde{0}=\widetilde{0}+\widetilde{a}=\widetilde{a} \end{array}\right] \quad ,\forall\widetilde{a},\widetilde{b},\widetilde{c}\in NF(R) \\ \widetilde{a}+(-\widetilde{a})=\widetilde{0}\Leftrightarrow a^s=a^d=0\Leftrightarrow\widetilde{a}=(a^m,0,0)\in R \end{array}\right] \quad (13)$$

The set of fuzzy numbers with the addition operation (**NF(R)**, +), makes an algebric structure of ***monoid additive commutative.***

***The real number associated*** to a fuzzy number $\tilde{a} = (a^m, a^M, a^s, a^d, \mu_{\tilde{a}})$, marked $< \tilde{a} > \in R$, is calculated with the following relations:

$$\left.\begin{aligned} &< \tilde{a} > \overset{\text{def.}}{=} \frac{a^m + a^M}{2} - As_{\tilde{a}} + Ad_{\tilde{a}} \\ &\text{where: } As_{\tilde{a}} = \int_{a^m - a^s}^{a^m} \mu_{\tilde{a}}(x)dx; \quad Ad_{\tilde{a}} = \int_{a^M}^{a^M + a^d} \mu_{\tilde{a}}(x)dx \end{aligned}\right] \qquad (14)$$

The two integrals represent the area of the surfaces between graph $\mu_{\tilde{a}}$ and the axes of the abscissas, areas situated to the left and respectively to the right of the nucleus.

With the help of the three notions (addition, multiplication with scalar and associated real number) one can defined the other operations as well as an order relation:

***a) substraction:*** $\tilde{a} - \tilde{b} \overset{\text{def.}}{=} \tilde{a} + (-\tilde{b}) = \tilde{a} + (-1)\cdot\tilde{b}$

***b) multiplication:*** $\tilde{a}\tilde{b} \overset{\text{def.}}{=} \dfrac{\tilde{a} < \tilde{b} > + < \tilde{a} > \tilde{b}}{2}$

***c) reverse:*** $\tilde{b}^{-1} \overset{\text{def.}}{=} \dfrac{\tilde{b}}{\langle \tilde{b} \rangle^2}$

***d) division:*** $\dfrac{\tilde{a}}{\tilde{b}} \overset{\text{def.}}{=} \tilde{a}\cdot\tilde{b}^{-1} = \dfrac{\tilde{a}\tilde{b}}{< \tilde{b} >^2} = \dfrac{\tilde{a} < \tilde{b} > + < \tilde{a} > \tilde{b}}{2 < \tilde{b} >^2}$ (15)

***e) degree increase:*** $\tilde{a}^n \overset{\text{def.}}{=} \tilde{a} < \tilde{a} >^{n-1}, \forall n \in Q$

***f) order relation:*** $\tilde{a} > \tilde{b} \Leftrightarrow < \tilde{a} > \; > \; < \tilde{b} >$

***g) resembling relation:*** $\tilde{a} \cong \tilde{b} \overset{\text{def.}}{\longleftrightarrow} < \tilde{a} > = < \tilde{b} >$

The definitions were conceived so that to preserve some important properties of the homologous operations with real numbers. Out of these one can remind:

$$\left.\begin{array}{l} <t\tilde{a}> = t<\tilde{a}>; \quad t(s\tilde{a}) = (ts)\tilde{a}; \quad t(\tilde{a}+\tilde{b}) = t\tilde{a}+t\tilde{b}; \\ <\tilde{a}\pm\tilde{b}> = <\tilde{a}> \pm <\tilde{b}>; \quad \left\langle \sum_{k=1}^{n} t_k \tilde{a}_k \right\rangle = \sum_{k=1}^{n} (t_k <\tilde{a}_k>); \\ <\tilde{a}\cdot\tilde{b}> = <\tilde{a}>\cdot<\tilde{b}>; \quad \left\langle \frac{\tilde{a}}{\tilde{b}} \right\rangle = \frac{<\tilde{a}>}{<\tilde{b}>}; \quad \left\langle \tilde{b}^{-1} \right\rangle = \left\langle \tilde{b} \right\rangle^{-1}; \\ \left\langle \sqrt[m]{\tilde{a}^n} \right\rangle = \sqrt[m]{<\tilde{a}>^n}; \\ \tilde{a}^n \tilde{a}^m = \tilde{a}^{n+m}; \; (\tilde{a}^n)^m = \tilde{a}^{nm}; \; \frac{\tilde{a}^n}{\tilde{a}^m} = \tilde{a}^{n-m}; \; (\tilde{a}\tilde{b})^n = \tilde{a}^n\tilde{b}^n; \; \left(\frac{\tilde{a}}{\tilde{b}}\right)^n = \frac{\tilde{a}^n}{\tilde{b}^n}. \end{array}\right] \quad (16)$$

The order relation defined in (15.f) is not a relation of total order.

Still, some very important properties of the order relation of the real numbers are true for the order relation with fuzzy numbers, too.

For instance:

$$\tilde{a} > \tilde{b} \quad \Rightarrow \quad \left[\begin{array}{ll} \tilde{a}\pm\tilde{c} > \tilde{b}\pm\tilde{c} \quad ,\forall\tilde{c} & \\ \tilde{a}\tilde{c} > \tilde{b}\tilde{c} \quad ,\forall\tilde{c} > 0 & \tilde{a}\tilde{c} < \tilde{b}\tilde{c} \quad ,\forall\tilde{c} < 0 \\ \dfrac{\tilde{a}}{\tilde{c}} > \dfrac{\tilde{b}}{\tilde{c}} \quad ,\forall\tilde{c} > 0 & \dfrac{\tilde{a}}{\tilde{c}} < \dfrac{\tilde{b}}{\tilde{c}} \quad ,\forall\tilde{c} < 0 \end{array}\right. \quad (17)$$

The resemblance relation defined in (15.g) is reflexive, symmetrical and transitive being in fact a relation of equivalence:

$$\tilde{a} \cong \tilde{a}; \quad \tilde{a} \cong \tilde{b} \Rightarrow \tilde{b} \cong \tilde{a}; \quad \left.\begin{array}{l} \tilde{a} \cong \tilde{b} \\ \tilde{b} \cong \tilde{c} \end{array}\right] \Rightarrow \tilde{a} \cong \tilde{c} \quad (18)$$

The space of this relation of equivalence $NF(R)/\cong$ is isomorphous with the set of real numbers:

$$\left.\begin{array}{l} NF(R)/\cong \; \overset{def.}{=} \; \{\hat{r}\}_{r\in R}; \quad \hat{r} \overset{def.}{=} \{\tilde{a} \in NF(R) \,|\, <\tilde{a}> = r\} \\ NF(R)/\cong \; \overset{f.bij.}{\longleftrightarrow} \; R \\ \bigcup_{r\in R} \hat{r} = NF(R); \quad \hat{r}_1 \cap \hat{r}_2 = \Phi \quad ,\forall r_1 \neq r_2 \end{array}\right] \quad (19)$$

Any fuzzy number of an equivalence class can be considered to be representative of the respective class, but, ussualy it is preferred as representative the pure real number r (unique) contained too, in the respective class.

Bijection f defining the isomorphism between the space as well as R preserves all the operations previously defined:

$$\left.\begin{array}{l} f(\hat{r}_1 \circ \hat{r}_2) = r_1 \circ r_2 \\ \hat{r}_1 > \hat{r}_2 \xleftrightarrow{\text{def.}} r_1 > r_2 \end{array}\right] \tag{20}$$

Therefore, the space as long as it has a complex algebric structure of total settled commutative body (and of vectorial space).

Taking into account that in applications the trapezoidal fuzzy numbers are preferred (and the sub-class of triangular fuzzy numbers) one will particularize the definitions of the addition and of the associated real number for these classes of fuzzy numbers:

$$\left.\begin{array}{l} \tilde{a} + \tilde{b} = (a^m + b^m, a^M + b^M, a^s + b^s, a^d + b^d) \\ < \tilde{a} > = \dfrac{a^m + a^M - a^s + a^d}{2} \end{array}\right] , \forall \tilde{a}, \tilde{b} \in \mathbf{Tp(R)} \tag{21}$$

$$\left.\begin{array}{l} \tilde{a} + \tilde{b} = (a^m + b^m, a^s + b^s, a^d + b^d) \\ < \tilde{a} > = a^m + \dfrac{a^d - a^s}{2} \end{array}\right] , \forall \tilde{a}, \tilde{b} \in \mathbf{Tr(R)} \tag{22}$$

Another great benefit of the way which the operations were introduced, is the fact that some ***internal operations*** on **Tp(R)** and respectively on **Tr(R)**.

## 4. Classic and fuzzy models

Formally, a ***classic model*** $M = (p, Z_p, A_p)$ is a third form made of the vector of the entrance parameters $p \in R^k$, optimized function (max/min) $Z_p : R^m \longrightarrow R$ and the system of admission conditions (made of a system of m equality and inequality with and unknown, usually non-linear): $A_p(x) \overset{\leq}{\underset{\geq}{=}} 0, x \in R^n$.

The set of admissible solutions and the set of optimal solutions are:

$$S^A = \left\{ x \in R^n \mid A_p(x) \overset{\geq}{\underset{\leq}{=}} 0 \right\} \text{ and } S^O = \left\{ x^\bullet \in R^n \mid Z_p(x^\bullet) \overset{\geq}{\underset{\leq}{}} Z_p(x), \forall x^\bullet \in S^A \right\}.$$

Two models $M_1$ and $M_2$ are ***the equivalent models*** if they have the same sets of admissible and optimal solutions: $M_1 \leftrightarrow M_2 \begin{cases} S_1^A = S_2^A \\ S_1^O = S_2^O \end{cases}$.

A ***fuzzy model*** $\widetilde{M} = (\widetilde{p}, Z_{\widetilde{p}}, A_{\widetilde{p}})$ is a model for which all the entrance parameters are fuzzy numbers, and to this one can associate a ***classic model*** $\langle \widetilde{M} \rangle = (\langle \widetilde{p} \rangle, Z_{\langle \widetilde{p} \rangle}, A_{\langle \widetilde{p} \rangle})$ which is obtained by replacing in the fuzzy model all fuzzy numbers with associated real numbers.

A very important result is the one given by the following theorem:

***Theorem*** (of fuzzy model charaterization).

***Any fuzzy model is equivalent with its associate model.***

The characterization of the fuzzy models allows as to precise the stages that have to be followed for solving a fuzzy model:

- the classic model is associated to the fuzzy model by replacing all fuzzy numbers with their associate real numbers;
- the classic model associated is solved by methods specific to any type of model. The literature in the field is very generous in this respect. The optimal solutions of this model (classic) are thus obtained;
- the optimal solution(s) determined in the previous stage is substituted in the function to be optimized (fuzzy) thus obtaining, besides the optimal solution(s), the optimal value, too: $\widetilde{v}^* = Z_{\widetilde{p}}(x^*)$.

An example for solving a fuzzy model is present in the article "*Determination of optimal ways in fuzzy graphs*" within the hereby publication.

## References


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