OVIDIU GHERASIM

A FUZZY APPROACH TO THE COMPETITION OF THE SUPPLIERS' STUDY

Classical matrix games

A **supplier** has the possibility to act in m modes, called strategies $S_f = \{s_1, s_2, ..., s_m\}$ in order to sell his products and obtain the biggest profit. In order to simplify the problem, we shall consider all the other suppliers as a single one which we shall name **partner**. The partner has, as well, n strategies $S_p = \{t_1, t_2, ..., t_n\}$ he can use no matter which strategy s_i is used by the supplier at a given moment.

When the two players (the supplier and the partner) use the strategies s_i and t_j, the real number $c_{ij} = f(s_i, t_j)$ represents the income earned by the supplier, respectively the loss suffered by the partner.

The function $f : S_f \times S_p \rightarrow \mathbf{R}$ praises in a unique way a matrix named the **income matrix** $C = (c_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$.

The triplet $\Gamma = (S_f, S_p, f)$ is named **matrix game** for two persons, finite, with null sum.

Table no. 1

The income matrix of a matrix game

	t_1	t_2	...	t_j	...	t_n
s_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}
s_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}
...	...	...	...	...	...	...
s_i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}
...	...	...	...	...	...	...
s_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}

An. Inst. cerc. ec. „Gh. Zane", t. 12, Iaşi, 2003, p. 35–45

During a determined period of time, the supplier uses the m strategies, alternatively, during certain small periods.

The vector formed by the probabilities with which the supplier uses the strategies $x=(x_1,x_2,\ldots,x_i,\ldots,x_m)$ is named **mixed strategy.** The partner will use as mixed strategies, vectors with n probability components $y=(y_1,y_2,\ldots,y_j,\ldots,y_n)$.

Because both mixed strategies components are probabilities, these fulfill the conditions:

$$\left.\begin{array}{l} \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0, \forall i = \overline{1,m} \\ \sum_{j=1}^{n} y_j = 1,\ y_j \geq 0, \forall j = \overline{1,n} \end{array}\right] \quad (1)$$

All the mixed strategies are S_f and S_p.

The medium income of the supplier, and the medium loss of the partner, when using the mixed strategies x and y is calculated by the relation:

$$F(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_i y_j \quad (2)$$

The mixed strategies **(x^*,y^*)** are ***optimum*** and **$v=F(x^*,y^*)$** is the ***game value*** if both strategies fulfill the condition:

$$F(x,y^*) \leq F(x^*,y^*) \leq F(x^*,y) \qquad ,\forall x \in S_F,\ y \in S_P \quad (3)$$

The problem of determining the optimum mixed strategies of the matrix games is completely solved, the specialty literature being very rich in this regard. Out of the works that debate the theory of games I have selected the books [2], [6] and [8] from the bibliography.

In fact, solving the matrix game stands is solving the next pair of dual linear programs:

$$\left.\begin{array}{c|c} [min]\sum_{i=1}^{m} x_i' & [max]\sum_{j=1}^{n} y_j' \\ \hline \sum_{i=1}^{m} c_{ij} x_i' \geq 1 \quad , j=\overline{1,n} & \sum_{j=1}^{n} c_{ij} y_j' \leq 1 \quad ,i=\overline{1,m} \\ x_{1\ldots m}' \geq 0 & y_{1\ldots n}' \geq 0 \end{array}\right] \quad (4)$$

If x'^* and y'^* are the best solutions of the two linear programs than the game values and the optimum mixed strategies are obtained by the relations:

$$\left.\begin{array}{l}\frac{1}{v}=\sum_{i=1}^{m} x_i^{**}=\sum_{j=1}^{n} y_j^{**} \\ x^*=(vx_1^{**},\dots,vx_m^{**}) \\ y^*=(vy_1^{**},\dots,vy_n^{**})\end{array}\right] \quad (5)$$

Fuzzy Matrix Games

Any economical problem, nowadays, is strongly affected by incertitude. In the article *A Fuzzy Approach in Establishing the Best Decisional Alternatives* from the present *Year-Book*, I presented the way the information can be mathematically modeled by using fuzzy numbers.

Other works on fuzzy theory whose subjects are strongly related to that of the present article are [1], [4] and [7].

Solving the linear programs that have fuzzy numbers as parameters is the main subject treated in the article [3]

I shall remind here only the definitions of the operations with fuzzy triangular numbers that will be used in the present article:

$\tilde{a}=(a_s, a_m, a_d) \quad \tilde{b}=(b_s, b_m, b_d)$		
associate real number:	$<\tilde{a}>=\frac{2a_m+a_s+a_d}{4}$	(6)
multiplication by a scalar:	$t\tilde{a}=\begin{cases}(ta_s, ta_m, ta_d) & ,t>0 \\ (ta_d, ta_m, ta_s) & ,t<0\end{cases}$	
addition:	$\tilde{a}+\tilde{b}=(a_s+b_s, a_m+b_m, a_d+b_d)$	
order relation:	$<\tilde{a}> \underset{(>)}{<} <\tilde{b}> \Rightarrow \tilde{a} \underset{(\succ)}{\prec} \tilde{b}$	

The multiplication by a scalar and the addition of fuzzy numbers are preserved to the associate real number:

$$\left.\begin{array}{l}<t\tilde{a}>=t<\tilde{a}> \\ <\tilde{a}+\tilde{b}>=<\tilde{a}>+<\tilde{b}>\end{array}\right] \quad (7)$$

A ***matrix fuzzy game*** $\tilde{\Gamma}$ is a game where the income matrix has fuzzy numbers as components:

$$\tilde{C}=(\tilde{c}_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \tag{8}$$

To a fuzzy matrix game can be associated a classic matrix game $\langle\tilde{\Gamma}\rangle$ who has real associate numbers as its elements:

$$\langle\tilde{C}\rangle=(<\tilde{c}_{ij}>)_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \tag{9}$$

Two ***matrix games*** are ***equivalent*** $\Gamma_1 \leftrightarrow \Gamma_2$ if they have the same best mixed strategies (but not necessarily equal values).

Theorem.

Any fuzzy matrix game is equivalent to his associate game.

Demonstration.

(x^*, y^*) are optimum mixed strategies for $\tilde{\Gamma} \Leftrightarrow$

$$\Leftrightarrow \sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}x_i y_j^* \le \sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}x_i^* y_j^* \le \sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}x_i^* y_j \quad ,\forall x\in S_F ,y\in S_P \Leftrightarrow$$

$$\Leftrightarrow \left\langle\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}x_i y_j^*\right\rangle \le \left\langle\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}x_i^* y_j^*\right\rangle \le \left\langle\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}x_i^* y_j\right\rangle \quad ,\forall x\in S_F ,y\in S_P \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=1}^{m}\sum_{j=1}^{n}<\tilde{c}_{ij}>x_i y_j^* \le \sum_{i=1}^{m}\sum_{j=1}^{n}<\tilde{c}_{ij}>x_i^* y_j^* \le \sum_{i=1}^{m}\sum_{j=1}^{n}<\tilde{c}_{ij}>x_i^* y_j \quad ,\forall x\in S_F ,y\in S_P \Leftrightarrow$$

$\Leftrightarrow$ (x^*, y^*) are optimum mixed strategies for $<\tilde{\Gamma}>$.

To highlight the work method, the calculations for a hypothetical matrix game of reduced dimensions are presented as follows.

The competition between two car providers

The competition between two car manufacturers companies *Auto1* and *Auto2*, of the same capacity and design, is the result of the need to gain high rates on the market on account of the competitor.

*Auto*1 (the supplier) uses two marketing strategies:

s_1 - advertising in all media

s_2 - TV advertising only.

Auto2 (the partner) uses three marketing strategies:

t_1 - radio advertising

t_2 - no advertising

t_3 - floods the market with cars at lower price

The six possible strategy averages of the two competitive companies and the market shares (%), earned by *Auto1* and lost by *Auto2*, calculated as averages of some polls, are those in table no.2.

Table no.2.

The income matrix of the initial game

	t_1	t_2	t_3
s_1	4	0	2
s_2	6	7	1

In order to determine the optimum mixed strategies the two linear dual programs correspondent to the matrix ***C*** must be solved:

$$[min]\ x_1' + x_2' \qquad \begin{cases} 4x_1' + 6x_2' \geq 1 \\ 7x_2' \geq 1 \\ 2x_1' + x_2' \geq 1 \end{cases} \qquad x_1' \geq 0,\ x_2' \geq 0$$

$$[max]\ y_1' + y_2' + y_3' \qquad \begin{cases} 4y_1' + 2y_3' \leq 1 \\ 6y_1' + 7y_2' + y_3' \leq 1 \end{cases} \qquad y_1' \geq 0,\ y_2' \geq 0,\ y_3' \geq 0 \qquad (10)$$

The first restriction can be eliminated from the primal program because it is obvious when the third restriction is fulfilled.

Its solving needs the introduction of two ecart variables and two artificial variables.

Table no.3

The simplex algorithm for the primal program

	x_1'	$\downarrow x_2'$	x_3'	x_4'	x_5'	x_6'		
z	1	1	0	0	0	0	0	
$\leftarrow x_5'$	0	[7]	–1	0	1	0	1	1/7<1
x_6'	2	1	0	–1	0	1	1	1/1
z'	–2	–8	1	1	0	0	–2	

Tabelul 3 (continued)

	$\downarrow x_1'$	x_2'	x_3'	x_4'	x_5'	x_6'		
z	1	0	1/7	0	–1/7	0	–1/7	
x_2'	0	1	–1/7	0	1/7	0	1/7	
$\leftarrow x_6'$	[2]	0	1/7	–1	–1/7	1	6/7	
z'	–2	0	–1/7	1	8/7	0	–6/7	
	x_1'	x_2'	x_3'	x_4'	x_5'	x_6'		
z	0	0	1/14	1/2	–1/14	–1/2	–4/7	
x_2'	0	1	–1/7	0	1/7	0	1/7	
x_1'	1	0	1/14	–1/2	–1/14	1/2	3/7	
z'	0	0	0	0	1	1	0	

The optimum solutions of the primal program are:

$$x_1'^{*}=\frac{3}{7},\quad x_2'^{*}=\frac{1}{7}.$$

The value of the game and the optimum mixed strategy of the supplier is thus calculated:

$$v=\frac{1}{\frac{4}{7}}=\frac{7}{4};\quad x_1^{*}=\frac{7}{4}\cdot\frac{3}{7}=\frac{3}{4};\qquad x_2^{*}=\frac{7}{4}\cdot\frac{1}{7}=\frac{1}{4};\qquad x^{*}=\left(\frac{1}{4},\frac{3}{4}\right).$$

Solving the dual program in the relations (10) needs only two ecart variables, the simple suitable algorithm going in the final phase.

Table no. 4.

The simplex algorithm for the dual

	$\downarrow y_1'$	y_2'	y_3'	y_4'	y_5'		
z	–1	–1	–1	0	0	0	
y_4'	4	0	2	1	0	1	1/4
$\leftarrow y_5'$	[6]	7	1	0	1	1	1/6
	y_1'	y_2'	$\downarrow y_3'$	y_4'	y_5'		
z	0	1/6	–5/6	0	1/6	1/6	

Tabelul 4 (continued)

$\leftarrow y_4'$	0	–14/3	[4/3]	1	–2/3	1/3	1/4
y_1'	1	7/6	1/6	0	1/6	1/6	1
	y_1'	$\downarrow y_2'$	y_3'	y_4'	y_5'		
z	0	–11/4	0	5/8	–1/4	3/8	
y_3'	0	–7/2	1	3/4	–1/2	1/4	–
$\leftarrow y_1'$	1	[7/4]	0	–1/8	1/4	1/8	2/7
	y_1'	y_2'	y_3'	y_4'	y_5'		
z	11/7	0	0	3/7	1	4/7	
y_3'	2	0	1	…	…	1/2	
y_2'	4/7	1	0	…	…	1/14	

The dual program has the optimum solutions $y_1'^{*} = 0,\ y_2'^{*} = \frac{1}{14}$, $y_3'^{*} = \frac{1}{2}$ the game value and the best mixed strategy of the partner are:

$$v = \frac{1}{\frac{4}{7}} = \frac{7}{4};\quad y_1^{*} = \frac{7}{4}\cdot 0 = 0;\quad y_2^{*} = \frac{7}{4}\cdot\frac{1}{14} = \frac{1}{8};\quad y_3^{*} = \frac{7}{4}\cdot\frac{1}{2} = \frac{7}{8};\quad y^{*} = \left(0, \frac{1}{8}, \frac{7}{8}\right)$$

Because the used tests have different errors, these errors shall be modeled using triangular fuzzy numbers obtaining thus the fuzzy matrix game I the following table:

Table no. 5.

The earning matrix for the fuzzy game

	t_1	t_2	t_3
s_1	<3; 4; 7>	<–1; 0; 1>	<1; 2; 5>
s_2	<3; 6; 7>	<3; 7; 9>	<0; 1; 4>

For each of the six fuzzy numbers associate real numbers will be calculated, using the first definition in the relations (6):

$$<3, 4, 7> \ = \frac{2\cdot 4+3+7}{4} = \frac{9}{2};\qquad <-1, 0, 1> \ = \frac{2\cdot 0-1+1}{4} = 0;$$

$$<1, 2, 5> \ = \frac{2\cdot 2+1+5}{4} = \frac{5}{2};\qquad <3, 6, 7> \ = \frac{2\cdot 6+3+7}{4} = \frac{11}{2};$$

$$<3, 7, 9> = \frac{2\cdot 7+3+9}{4}=\frac{13}{2}; \qquad <0, 1, 4> = \frac{2\cdot 1+0+4}{4}=\frac{3}{2}.$$

The classical associate game is the one in the following table:

Table no. 6.

The earning matrix for the associate game

	t_1	t_2	t_3
s_1	$\frac{9}{2}$	0	$\frac{5}{2}$
s_2	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{3}{2}$

In order to solve the fuzzy game the classical associate game will be solved and the solution (the pair of optimum mixed strategies) will be used in the fuzzy game.

Thus, the two linear dual programs are associated to the classical game:

$$\begin{array}{c|c} [min]\ x_1' + x_2' & [max]\ y_1' + y_2' + y_3' \\ \hline \begin{cases} \frac{13}{2}x_2' \geq 1 \\ \frac{5}{2}x_1' + \frac{3}{2}x_2' \geq 1 \end{cases} & \begin{cases} \frac{9}{2}y_1' + \frac{5}{2}y_3' \leq 1 \\ \frac{11}{2}y_1' + \frac{13}{2}y_2' + \frac{3}{2}y_3' \leq 1 \end{cases} \\ x_1' \geq 0,\ x_2' \geq 0 & y_1' \geq 0,\ y_2' \geq 0,\ y_3' \geq 0 \end{array} \qquad (11)$$

The fractions can be eliminated by doubling all the restrictions:

$$\begin{array}{c|c} [min]\ x_1' + x_2' & [max]\ y_1' + y_2' + y_3' \\ \hline \begin{cases} 13x_2' \geq 2 \\ 5x_1' + 3x_2' \geq 2 \end{cases} & \begin{cases} 9y_1' + 5y_3' \leq 2 \\ 11y_1' + 13y_2' + 3y_3' \leq 2 \end{cases} \\ x_1' \geq 0,\ x_2' \geq 0 & y_1' \geq 0,\ y_2' \geq 0,\ y_3' \geq 0 \end{array} \qquad (12)$$

The solving of the primal program is given in table no.7 and that of the dual program in table no.8.

The optimum solutions of the two linear programs are:

$$x_1'^{*} = \frac{4}{13}; \quad x_2'^{*} = \frac{2}{13}; \quad y_1'^{*} = 0; \quad y_2'^{*} = \frac{4}{65}; \quad y_3'^{*} = \frac{2}{5}.$$

The value of the game and the suitable optimum mixed strategies are:

$$v = \frac{1}{\frac{6}{13}} = \frac{13}{6}; \quad x_1^* = \frac{13}{6} \cdot \frac{4}{13} = \frac{2}{3}; \quad x_2^* = \frac{13}{6} \cdot \frac{2}{13} = \frac{1}{3}; \quad x^* = \left(\frac{2}{3}; \frac{1}{3}\right);$$

$$y_1^* = \frac{13}{6} \cdot 0 = 0; \quad y_2^* = \frac{13}{6} \cdot \frac{4}{65} = \frac{2}{15}; \quad y_3^* = \frac{13}{6} \cdot \frac{2}{5} = \frac{13}{15}; \quad y^* = \left(0; \frac{2}{15}; \frac{13}{15}\right).$$

Table no. 7.

The simplex algorithm for the primal program Phase
Phase I

	x_1'	$\downarrow x_2'$	x_3'	x_4'	x_5'	x_6'		
z	1	1	0	0	0	0	0	
$\leftarrow x_5'$	0	[13]	–1	0	1	0	2	2/13
x_6'	5	3	0	–1	0	1	2	2/3
z'	–5	–16	1	1	0	0	–4	
	$\downarrow x_1'$	x_2'	x_3'	x_4'	x_5'	x_6'		
z	1	0	1/13	0	–1/13	0	–2/13	
x_2'	0	1	–1/13	0	1/13	0	2/13	–
$\leftarrow x_6'$	[5]	0	3/13	–1	–3/13	1	20/13	4/13
z'	–5	0	–3/13	1	16/13	0	–20/13	
	x_1'	x_2'	x_3'	x_4'	x_5'	x_6'		
z	0	0	2/65	1/5	–2/65	–1/5	–6/13	
x_2'	0	1	–1/13	0	1/13	0	2/13	
x_1'	1	0	3/65	–1/5	–3/65	1/5	4/13	
z'	0	0	0	0	1	1	0	

Table no. 8.

The simplex algorithm for the dual program

	y_1	y_2	$\downarrow y_3$	y_4	y_5	
z	–1	–1	–1	0	0	0
$\leftarrow y_4'$	9	0	[5]	1	0	2
y_5	11	13	3	0	1	2

Tabelul 8 (continued)

	y_1'	$\downarrow y_2'$	y_3'	y_4'	y_5'		
z	4/5	–1	0	1/5	0	2/5	
y_1'	9/5	0	1	1/5	0	2/5	
$\leftarrow y_5'$	28/5	13	0	–3/5	1	4/5	
	y_1'	y_2'	$\downarrow y_3'$	y_4'	y_5'		
z	16/3	0	0	16/65	1/13	6/13	
$\leftarrow y_1'$	9/5	0	1	1/5	0	2/5	
y_2'	28/65	1	0	–3/65	1/13	4/65	

The fuzzy game has the same optimum mixed strategies and its value is a fuzzy triangular number thus calculated:

$$\tilde{v} = F(x, y) = \sum_{i=1}^{2} \sum_{j=1}^{3} \tilde{c}_{ij} x_i^* y_j^* =$$

$$= \frac{2 \cdot 0}{3}(3;\ 4;\ 7) + \frac{2 \cdot 2}{3 \cdot 15}(-1;\ 0;\ 1) + \frac{2 \cdot 13}{3 \cdot 15}(1;\ 2;\ 5) + \frac{1 \cdot 0}{3}(3;\ 6;\ 7) + \frac{1 \cdot 2}{3 \cdot 15}(3;\ 7;\ 9) +$$

$$+ \frac{1 \cdot 13}{3 \cdot 15}(0;\ 1;\ 4) = \frac{(-4;\ 0;\ 4) + (26;\ 52;\ 130) + (6;\ 14;\ 18) + (0;\ 13;\ 52)}{45} =$$

$$= \frac{(28;\ 79;\ 204)}{45} \approx \left(0.6(2);\ 1.7(5);\ 4.5(3)\right).$$

$$< \tilde{v} > = \frac{2 \cdot 79 + 28 + 204}{45} = \frac{39}{18} = 2.1(6).$$

Conclusions

Although the game from the presented example is hypothetical and is of reduced dimensions, the results are extremely important.

Therefore, the initial game, where the modeling of the doubting entrance information was not taken into account, had as a result a mixed optimum strategy for *Auto1* where the pure strategy s_2 is preferred (75%). After modeling the incertitude, by solving the fuzzy game, *Auto1* totally changed its options, the optimum mixed strategy preferred now being the pure strategy s_1 (67%).

The conclusion is obvious. If a manager does not take into consideration the incertitude modeling by using the fuzzy techniques, his decisions will be slightly insufficient supported, and therefore, inefficient.

References

1. Bojadziev G., Bojadziev M., *Fuzzy Sets, Fuzzy Logic, Applications*, London, 1995.
2. Ciucu G., Iosifescu M., *Teoria jocurilor*, Ed. Tehnică, Bucureşti, 1965.
3. Gherasim O, *Fundamentarea deciziilor de tip programare liniară prin sisteme fuzzy*, Anuarul Institutului de Cercetări Economice „Gh. Zane", nr.8-9/1999-2000, Ed. Academiei Române, Iaşi.
4. Gherasim O, *Modelarea incertitudinii prin numere fuzzy. Aritmetica numerelor fuzzy*, Anuarul Institutului de Cercetări Economice şi Sociale „Gh. Zane", nr.12/2003, Ed. Academiei Române, Iaşi, 2003.
5. Gherasim O., *Studiul deciziilor de management prin sisteme fuzzy – Teză de doctorat*, Iaşi, 2003.
6. Karlin S., *Mathematical Methods and Theory of Games. Programming and Economics*, 2 vol., London, Pergamon Press, 1959.
7. Kaufmann, A., *Introduction à la théorie des sous-ensembles flous (Fuzzy sets theory)*, Ed. Masson et C[ie], Paris, 1973.
8. Owen G., Teoria jocurilor, Ed. Tehnică, Bucureşti, 1974.